Summation Notation

Jacco Thijssen

1 Introduction

In mathematics and statistics one often has to take the sum over a number of elements. For example, you might want to compute the sum of the first 10 natural numbers. You could write this, obviously, as

$$1+2+3+4+5+6+7+8+9+10=55.$$

If the number of elements to be added is large, writing every element down becomes rather cumbersome. Therefore, we often write this sum as

$$1+2+\cdots+10=55.$$

An even more elegant way to write it uses a standard mathematical notation, namely the Greek capital letter "Sigma" (S for "sum"):

$$\sum_{i=1}^{10} i = 55.$$

You should read this as: "The sum of all i, where i runs from 1 to 10". The symbol i is called the "summation index". The summation index is by no means restricted to i, as we will see later.

2 General Notation

Suppose that I have a sequence of elements $x_1, x_2, \ldots, x_n$. Each x_i could represent – literally – anything. In statistics, for example, suppose I have a sample containing n observations on Leaving Cert points of Junior Freshman BESS students. In that case, x_i denotes the leaving cert score for student i in my sample. Often we want to compute the average value, which we denote by $\bar{x}$.

In order to do so, we need the sum of all observations and divide them by the number of observations.

A convenient way of writing the sum is by using the Σ notation:[1]

$$\sum_{i=1}^{n} x_i := x_1 + x_2 + \cdots + x_n.$$

This should be read as: "the sum of all values x_i, where i runs from 1 to n". So, the average score of the sample of JF BESS students' leaving cert scores can now be written as:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{x_1 + \cdots + x_n}{n}.$$

As I said before, you do not have to use i as the summation index. By convention, we use i as the first choice, but there is nobody stopping us from using j, or k, or l, or a, or..... In fact,

$$\sum_{i=1}^{n} x_i = \sum_{j=1}^{n} x_j = \sum_{a=1}^{n} x_a.$$

The summation index is just there to keep track of what elements we are taking the summation of. The only thing that you have to keep in mind is that you have to be *consistent*. For, example, you cannot say "I am going to sum over j" and then use k as a subscript, i.e.

$$\sum_{j=1}^{n} x_k,$$

does not really mean anything.

Summation notation is particularly useful if you want to sum over elements of a *set*. For example, suppose that $S = \{(H,H),(T,T),(H,T),(T,H)\}$. We will encounter this set in probability theory. It represents the list of all possible outcomes of two coin flips: (Heads,Heads), (Tails,Tails), (Heads,Tails), and (Tails,Heads). Suppose that you are engaged in a bet involving two coin flips: you win EUR 1 each time Heads comes up, and you lose EUR 1 if Tails comes up. Denote the amount won by X. Then X is a function that maps the list

[1] The notation ":=" means: "is by definition equal to".

of all possible outcomes, S, to the set $\{-2, 0, 2\}$, since

$$X(s) = \begin{cases} -2 & \text{if } s = (T, T), \\ 0 & \text{if } s = (H, T), \text{ or } s = (T, H), \\ 2 & \text{if } s = (H, H). \end{cases}$$

The sum of all possible payoffs can now be written as

$$\sum_{s \in S} X(s) := X(T, T) + X(H, T) + X(T, H) = X(H, H) = 0,$$

which should be read as "the sum over all values $X(s)$, where s is some element of the set S". Note that here the summation index is an element s of the set S, i.e. a pair of outcomes (s_1, s_2).

In addition, you can write down the sum over all elements of a *subset* of a set. Suppose that I want to write down the sum of the payoffs of all pairs of coin flips which show Heads for the first flip. That is, I am interested in the subset $A = \{s \in S : s_1 = H\}$. this should be read as: "the set containing all elements of S for which the first element is H". In other words, $A = \{(H, H), (H, T)\}$. So,

$$\sum_{\{s \in S : s_1 = H\}} X(s) = X(H, H) + X(H, T) = 2.$$

Rules for summation notation are straightforward extensions of well-known properties of summation. For example,

$$\begin{aligned} \sum_{i=1}^{n} a x_i &= a x_1 + a x_2 + \cdots + a x_n \\ &= a(x_1 + x_2 + \cdots + x_n) \\ &= a \sum_{i=1}^{n} x_i. \end{aligned}$$

In other words, you can take a constant "out of the summation". This is nothing more than taking a constant out of brackets.

3 Double Summation

Sometimes data are grouped in a table (or matrix) like the one below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

This table has 2 rows and 3 columns and is therefore called a 2×3 matrix (to be read as "a 2-by-3 matrix"). Suppose that the numbers in the matrix represent a characteristic x. Each element of this matrix can then be represented as x_{ij}, where i indicates the row and j indicates the column. For example, $x_{23} = 9$, and $x_{12} = 2$. Suppose that I want to sum all elements in the matrix. Then I could sum all rows per column, or all columns per row, i.e.

$$\begin{aligned}\sum_{j=1}^{3}\sum_{i=1}^{2} x_{ij} &= (x_{11} + x_{21}) + (x_{12} + x_{22}) + (x_{13} + x_{23}) \\ &= (1+4) + (2+5) + (3+6) = 21,\end{aligned}$$

or

$$\begin{aligned}\sum_{i=1}^{2}\sum_{j=1}^{3} x_{ij} &= (x_{11} + x_{12} + x_{13}) + (x_{21} + x_{22} + x_{23}) \\ &= (1+2+3) + (4+5+6) = 21,\end{aligned}$$

respectively. Obviously, in the general case it holds that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} x_{ij},$$

as the order of summation does not matter.